Einstein's Hole Argument

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Abstract

In general relativity, a spacetime and a gravitational field form an indivisible unit: no field, no spacetime. This is a lesson of Einstein's *hole argument*. We use a simple transformation in a Schwartzschild spacetime to illustrate this.

> On the basis of the general theory of relativity ... space as opposed to "what fills space" ... has no separate existence. ... There is no such thing as an empty space, i.e., a space without [a gravitational] field. ... Space-time does not claim existence on its own, but only as a structural quality of the field.
>
> Albert Einstein, 1952.

Introduction. What justification can be given for Einstein's words,[1] written late in his life? The answer to this question has its origin in 1913, when Einstein was searching for a field equation for gravity. Einstein was aware of the possibility of generally covariant field equations, but he believed – wrongly, it turned out – that they could not possess the correct Newtonian limit. He then proposed a field equation covariant only under linear coordinate transformations. To buttress his case against generally covariant field equations, Einstein devised his *hole argument*, which purported to show that no generally covariant field equation can be satisfactory. When Einstein discovered his fully satisfactory generally covariant field equation for general relativity in 1915, it became apparent that there is a hole in the argument.[2]

The twin and pole-in-the-barn "paradoxes" of special relativity[3] are invalid arguments whose elucidation helps us better understand the theory. So too it is with the hole argument and general relativity. Einstein's words are lessons of the hole argument.

Many authors have written about the hole argument, mostly from an historical or philosophical perspective.[4] But none provide a simple concise account of the argument, its rebuttal, and its lessons. My purpose here is to provide such an account.

Other authors use a general spacetime in the hole argument. We use the Schwartzschild spacetime, whose simple geometry makes the argument more concrete and visualizable. This is sufficient to understand the argument and its lessons.

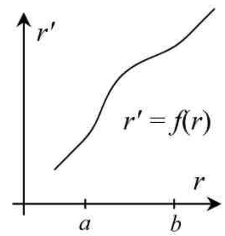

Fig. 1: The coordinate change $r' = f(r)$: $r' = r$ except in the hole $a < r < b$.

The Solutions G and G′. In general relativity, the gravitational field of a spherically symmetric central mass is usually represented by the Schwartzschild solution

$$G(r):\; ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 \;-\; \left(1 - \frac{2m}{r}\right)^{-1} dr^2 \;-\; r^2 d\Omega^2,$$

where $d\Omega^2 = \sin^2\varphi \, d\theta^2 + d\varphi^2$ is the metric of the unit 2-sphere, of area 4π. (We assume that the radius r_c of the central object is greater than the Schwartzschild radius $2m$. The solution is then valid for $r > r_c$.)

Define a coordinate change $r = f(r')$, where $r' = r$ for $r \notin (a, b)$, while $r' \neq r$ for $r \in (a, b)$ (the hole). Do not change the other coordinates. See Figure 1. Then $dr = f'(r')\, dr'$, and ds^2 in the new coordinate system is

$$G'(r'):\; ds^2 = \left(1 - \frac{2m}{f(r')}\right) dt^2 \;-\; \left(1 - \frac{2m}{f(r')}\right)^{-1} f'^2(r')\, dr'^2 \;-\; f^2(r')\, d\Omega^2.$$

The coordinate change is simply a relabeling of the events in the spacetime.

As the vacuum field equation is generally covariant, $G'(r')$ is, with $G(r)$, a solution. $G(r)$ and $G'(r')$ represent the same gravitational field on the same spacetime manifold of events. They are physically equivalent. For example, the sphere at r' in $G'(r')$ has area $4\pi f^2(r') = 4\pi r^2$, as does the sphere at r in $G(r)$.

Replace r' with r in $G'(r')$ to obtain

$$G'(r):\; ds^2 = \left(1 - \frac{2m}{f(r)}\right) dt^2 \;-\; \left(1 - \frac{2m}{f(r)}\right)^{-1} f'^2(r)\, dr^2 \;-\; f^2(r)\, d\Omega^2.$$

$G'(r)$ is a solution to the vacuum field equation, as it has the same mathematical form as the solution $G'(r')$.

Henceforth we shall be interested only in $G(r)$ and $G'(r)$, which we abbreviate to G and G'.

The Hole Argument. Under G, the sphere at r has area $4\pi r^2$; under G' it has area $4\pi f^2(r)$. Thus the solutions G and G' are physically distinguishable.

The two solutions show that the field equation does not uniquely determine the gravitational field of the central mass. The field equation is therefore unsatisfactory. Indeed, the argument shows that *no* generally covariant field equation can be satisfactory. In short,

The field equation of general relativity is unsatisfactory.

Resolution. The hole argument starts with two gravitational fields, G and G', on the same spacetime and using the same coordinates. Its conclusion, that no generally covariant field equation can be satisfactory, follows directly from this. Thus the hole argument only shows that *if* a theory allows two gravitational fields on the same spacetime, then it does not have a generally covariant field equation. Turn this around: If a theory has a generally covariant field equation, then it cannot consider two gravitational fields on the same spacetime.

In general relativity a spacetime manifold and a gravitational field form an indivisible unit. We are not free to start with a spacetime and then put a gravitational field on it. If we could, we could put a second field on the spacetime and suffer the hole argument. In short: no field, no spacetime.

This justifies Einstein's words at the start of this note. It is remarkable that such a deep result can be obtained from such simple considerations.

General relativity. General relativity considers G and G' to live on different spacetime manifolds, say M and N. To distinguish them, rename their r coordinates to r_M and r_N. Map an event $E_N \in N$ at r_N to the event $E_M \in M$ with $r_M = f(r_N)$, with the other coordinates unchanged. Then the transformation maps G' to G. For example, the radial term transforms properly:

$$\left(1 - \frac{2m}{r_M}\right)^{-1} dr_M^2 = \left(1 - \frac{2m}{f(r_N)}\right)^{-1} (f(r_N)\, dr_N)^2.$$

Under the physical correspondence $E_M \Leftrightarrow E_N$, all predictions of general relativity are the same for G and G'. For example, E_N is on a sphere of area $4\pi f^2(r_N)$ and E_M is on a sphere of area $4\pi r_M^2 = 4\pi f^2(r_N)$. And the field equation at E_N maps to the field equation at E_M, geodesics through E_N map to geodesics through E_M, etc. Thus in general relativity G and G' are physically indistinguishable.

We emphasize that we have been concerned here only with the concept of spacetime in general relativity. We are not concerned with the "real" nature of spacetime – even if that is a meaningful term. Just as Newtonian physics, special relativity, and general relativity have radically different concepts of spacetime, some future theory might have a radically different concept of spacetime from general relativity.

Applications. The hole argument is of intrinsic interest for what it tells us about the nature of spacetime in general relativity. Its lessons are also important in some applications of the theory. We discuss two.

1. The global positioning system (GPS) must take into account effects of general relativity to function properly. The GPS thus models the Earth's gravitational field with a spacetime metric. The metric "defines not only the gravitational field that is assumed, but also the coordinate system in which it is presented. There is no other source of information about the coordinates apart from the expression for the metric. It is also not possible to define the coordinate system in any way that does not require a unique expression for the metric. In most cases where the coordinates are chosen for computational convenience, the expression for the metric is the most efficient way to communicate clearly the choice of coordinates that is being made."[5]

The Schwartzschild solution $G(r)$ illustrates this. The physical meaning of the r coordinate can be read from the $r^2 d\Omega^2$ term: events on the sphere of area $4\pi r^2$ have radial coordinate r. The physical meaning of the t coordinate can also be read off from $G(r)$: for a clock at fixed r, θ, and ϕ, $ds^2 = (1 - 2m/r)dt^2$. This defines the coordinate time dt in terms of the time ds measured by the clock. The physical meaning of the r and t coordinates in $G'(r)$ is different.

2. The issues discussed here complicate the initial value problem for the field equations of general relativity.[6,7] Consider first, for comparison, the initial value problem for the source free Maxwell equations. The equations are of first order. Initial data consist of $\mathbf{E}$ and $\mathbf{B}$ fields at an initial time t_0. At t_0 the fields must satisfy the source free Maxwell equations not containing a time derivative: $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$. Then there is a unique solution to all of the Maxwell equations for $t \geq t_0$ which agrees with the initial data. The solution uses coordinates (t, x, y, z) on a flat spacetime manifold. The coordinates and the manifold are given in advance.

Now consider the vacuum field equations of general relativity. They are of second order. Thus one might expect that data consisting of the solution and its first derivatives on an initial spacelike 3-manifold, with some constraints from the field equations, would uniquely determine the solution at later times. But no such data can do this: any solution can be transformed to others by coordinate transformations which leave the data fixed. The field equations cannot even uniquely determine the topology of a manifold on which a solution is defined.[8]

Acknowledgment. I thank Martin Barrett for helpful comments.

[1]A. Einstein, *Relativity: The Special and the General Theory* (Methuen, London, 1952), p. 155.

[2]For an account of the history of general relativity, see R. Torretti, *Relativity and Geometry* (Pergamon Press, Oxford, 1983).

[3]E. Taylor and J. Wheeler, *Spacetime Physics* (Freeman, San Francisco, 1966).

[4]R. Torretti, Ref. 2; J. Norton, *How Einstein found his field equations: 1912-1915*, St. Hist. Phil. Sci. **14**, 253-316 (1984); J. Stachel, *What a physicist can learn from the History of Einstein's Discovery of General Relativity*, in *Proceedings of the Fourth Marcel Grossman Meeting on General Relativity*, Ed. R. Ruffini, (Elsevier, Amsterdam, 1986), pp. 1857-1862. J. Norton, *Einstein, the hole argument, and the reality of space*, in *Measurement, Realism and Objectivity*, Ed. J. Forge, (Dordrecht, Boston, 1987), pp. 153-188; J. Earman and J. Norton, *What Price Substantivalism? The Hole Story*, Brit. J. Phil. Sci. **38**, 515-525 (1987); J. Stachel, *Einstein's search for General Covariance, 1912-1915*, in *Einstein and the history of general relativity*, Eds. D. Howard and J. Stachel (Birkhäuser, Boston, 1989); J. Butterfield, *The Hole Truth*, Brit. J. Phil. Sci. **40**, 1-28 (1989); J. Earman, *World enough and space-time* (MIT Press, Cambridge, 1989); J. Stachel, *The Meaning of General Covariance*, in *Philosophical problems of the internal and external worlds*, Ed. J. Earman (University of Pittsburgh, Pittsburgh, 1993); R. Rynasiewicz, *The Lessons of the Hole Argument*, Brit. J. Phil. Sci. **45**, 407-436 (1994).

John Stachel revived interest in the hole argument by pointing out its subtleties. (Stachel, 1989. The paper was written in 1980 (Stachel, 1993).) Earlier commentators did not fully realize the argument's significance, because they concluded that Einstein made a simple mistake in it.

[5]C. Misner, *Precis of General Relativity*, gr-qc/9508043.

[6]S. Adler, M. Bazin, M. Schiffer, *Introduction to General Relativity* (McGraw-Hill, New York, 1965), Chapter 9.

[7]R. Hawking and R. Ellis, *The Large Scale Structure of Space-time* (Cambridge University Press, Cambridge, 1973), Chapter 7.

[8]C. Misner, K. Thorne, J. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), p. 837.